1 | 2 pt

Consider the problem

$$-\frac{\partial}{\partial x}\bar{u}u - \frac{\partial}{\partial y}\bar{v}u + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial}{\partial}\right.$$

$$= 0$$

where $\bar{u}(x,y) \equiv 1$
$\bar{v}(x,y) = 1 + x$
$\mu(x,y) = 1 + xy$

with boundary conditions

$$u(0,y) = 1, \quad u(x,0) = \cos x$$
$$\frac{\partial u}{\partial x}(1,y) = 0, \quad \frac{\partial u}{\partial y}(x,1) = 0$$

second-order accurate

1½ a Give the Finite volume discretization of the field equation on an equidistant grid with mesh size h in both directions.

Answer

First write it in divergence form

$$-\operatorname{div}\begin{bmatrix}\bar{u}u\\ \bar{v}u\end{bmatrix} + \operatorname{div}\begin{bmatrix}\mu\frac{\partial u}{\partial x} & -\frac{1}{2}\mu\frac{\partial u}{\partial y}\\ -\frac{1}{2}\mu\frac{\partial u}{\partial x} & \mu\frac{\partial u}{\partial y}\end{bmatrix} = 0$$

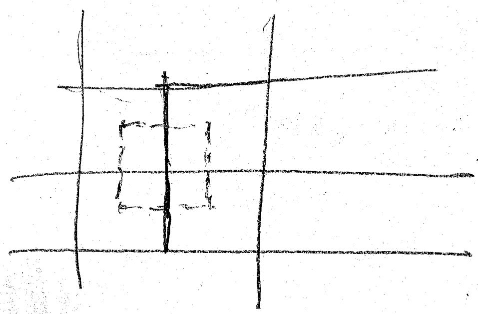
$$\rightarrow q = -\begin{bmatrix}\bar{u}u\\ \bar{v}u\end{bmatrix} + \begin{bmatrix} & \end{bmatrix}$$

$$\operatorname{div} q = 0$$

Gauss $\int_\Omega \operatorname{div} q \, d\Omega = \int_\Gamma q \cdot n \, d\Gamma$

$\frac{1}{2}$

1 2



$$h\left(q^{(1)}_{i+\frac{1}{2},j} - q^{(1)}_{i-\frac{1}{2},j}\right) + h\left(q^{(2)}_{i,j+\frac{1}{2}} - q^{(2)}_{i,j-\frac{1}{2}}\right) = 0$$

$$q^{(1)}_{i+\frac{1}{2},j} = \otimes + \mu(x_{i+\frac{1}{2}}, y_j)\left[\frac{u_{i+1,j} - u_{i,j}}{h} + \frac{1}{2}\,\frac{u_{i+1,j+1} - u_{i+1,j-1} + u_{i,j+1} - u_{i,j-1}}{4h}\right]$$

$$q^{(2)}_{i,j+\frac{1}{2}} = \oplus + \mu(x_i, y_{j+\frac{1}{2}})\left[\frac{1}{2}\,\frac{u_{i+1,j+1} - u_{i+1,j-1} + u_{i,j+1} - u_{i,j-1}}{4h} + \frac{u_{i,j+1} - u_{i,j}}{h}\right]$$

$$\otimes = -\bar{u}(x_{i+\frac{1}{2}}, y_j)(u_{i+1,j} + u_{i,j})/2$$

$$\oplus = -\bar{v}(x_i, y_{j+\frac{1}{2}})(u_{i,j+1} + u_{i,j})/2$$

$$\mu(x_{i+\frac{1}{2}}, y_j) = 1 + x_{i+\frac{1}{2}} y_j, \quad \mu(x_i, y_{j+\frac{1}{2}}) = 1 + x_i \cdot y_{j+\frac{1}{2}}$$

$$\bar{u}(x_{i+\frac{1}{2}}, y_j) = 1$$

$$\bar{v}(x_i, y_{j+\frac{1}{2}}) = 1 + x_i$$

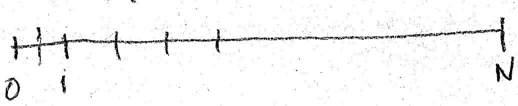
0.7

0.3

0.5pt

3 b Give the discretization of the boundary conditions when the Dirichlet is imposed at the interface of the control volume and the Neumann condition at the grid point. Also give the mesh size that results from this positioning. and the x,y location of the grid points, and the i,j for which the field equation are checked

Answer.



$\rightarrow (N-\frac{1}{2})h = 1$ $\qquad x_i = (i-\frac{1}{2})h$, $y_j = (j-\frac{1}{2})h$

$(u_{0,j}+u_{1,j})/2 = 1$, $(u_{i,0}+u_{i,1})/2 = \cos(x_i)$ 0.25

$u_{N+1,j} - u_{N-1,j} = 0$, $u_{i,N+1} - u_{i,N-1} = 0$ 0.25

$1 \le i,j \le N$

c ?

~~Show that~~

$$\cancel{\sum_{j=1}^{N} h(q_{N+\frac{1}{2},j} - q_{\frac{1}{2},j})} + \sum_{i=1}^{N} h(q_{i,N+\frac{1}{2}} - q_{i,\frac{1}{2}}) = 0$$

2.1 Consider the discretization of $\bar{u}\frac{du}{dx} + \mu\frac{d^2u}{dx^2} = 0$ $\frac{du}{dx}(0) = 5$

given by $\bar{u}\frac{u_{j+1} - u_j}{h} + \mu\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0$

for $j = \phi, \ldots, N-1$

$u_N = 1$, ~~$u_j(0) = 0$ for $j = 0, \ldots, N-1$~~ $h = 1/N - \frac{1}{2}$

$x_j = -\frac{1}{2}h + jh$

$\bar{u}\frac{u_1 - u_0}{h} + \mu\left(\frac{u_1 - u_0}{h^2} + \frac{5}{h}\right) = 0$ $x_0 = 0 \pm h$, $x_N = 1$

a For which combination of $\bar{u}, \mu, h$ will the solution be monotonous

Answer

$$\left(\frac{\bar{u}}{h} + \frac{\mu}{h^2}\right)(u_{j+1} - u_j) = \frac{\mu}{h^2}(u_j - u_{j-1})$$

$$\left(\frac{\bar{u}h}{\mu} + 1\right)(u_{j+1} - u_j) = u_j - u_{j-1}$$

Monotonous if $\frac{\bar{u}h}{\mu} + 1 > 0 \rightarrow \frac{\bar{u}h}{\mu} > -1$

0.5

0.5

b In which case is there a restriction on h in order to have

Answer if $\frac{\bar{u}}{\mu} > 0$ then no restriction

if $\frac{\bar{u}}{\mu} < 0$ then $h < -\frac{\mu}{\bar{u}}$

0.5

or if they to $\bar{u}h > -4$ no restriction for $\bar{u} \geq 0$

$h < -\frac{4}{\bar{u}}$ if $\bar{u} < 0$

2_2

2c

Give a discretization of the continuous equation for a non-equidistant grid. Where the grid points are given by a function

$$x = g(\xi)$$

where $g(\xi)$ is a monotonously increasing function mapping the interval $[0,1]$ to $[0,1]$.
Make sure that for the choice $g(t)=t$ we get back the above discretization

$$\bar{u}\,\frac{u_{j+1}-u_j}{x_{j+1}-x_j} + \mu\left[\frac{2u_{j+1}}{(x_{j+1}-x_j)(x_{j+1}-x_{j-1})} - \frac{2u_j}{(x_{j+1}-x_j)(x_j-x_{j-1})} + \frac{2u_{j-1}}{(x_j-x_{j-1})(x_{j+1}-x_{j-1})}\right] = 0$$

0.4

$$u_N = 1$$

$$\frac{du}{dx} = 5 \rightarrow \frac{u_1-u_0}{x_1-x_0} = 5 \rightarrow u_0 = u_1 - 5(x_1-x_0)$$

$$\bar{u}\,\frac{u_2-u_1}{x_2-x_1} + \mu\left[\frac{2u_2}{(x_2-x_1)(x_2-x_0)} - \frac{2u_1}{(x_2-x_1)(x_1-x_0)} + \frac{2u_1}{(x_1-x_0)(x_2-x_0)} - \frac{10}{x_2-x_0}\right] = 0$$

0.4

0.2

$$x_j = g(\xi_j)$$
$$\xi_j = (j-\tfrac{1}{2})h, \quad h = 1/N-\tfrac{1}{2}$$

Check this for

3. $9\frac{1}{2}$ pts

Consider the equation

$$e^{-x}\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\cos(x)\frac{\partial u}{\partial x}\right), \quad u(0,t) = 1, \; u(1,t) = 0 \quad \text{on } (0,1)$$

$$u(x,0) = 0$$

a Transform this equation into a system of ODEs by a finite element discretization where we have a set of basis functions $\phi_1, \dots \phi_N \in H^1$ satisfying $\phi(0)=\phi(1)=0$

$1\frac{1}{2}$ pts

$$M\frac{dc}{dt} = Ac + b$$

Answer

First make boundary condition homogeneous.

E.g. take $\bar{u}(x,t) = 1 - x$ 0.3

Substitute $u = \bar{u} + \tilde{u}$

$$\rightarrow e^{-x}\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial x}\left(\cos(x)\frac{\partial \tilde{u}}{\partial x}\right) + \frac{\partial}{\partial x}\cos x, \quad \tilde{u}(0,t) = \tilde{u}(1,t) = 0$$

$$\tilde{u}(x,0) = -\bar{u} = x - 1$$

0.1

$$r_1(u) = \left(e^{-x}\frac{\partial \tilde{u}}{\partial t} - \frac{\partial}{\partial x}\left(\cos x \frac{\partial \tilde{u}}{\partial x}\right) + \sin x\right) \text{ for } t>0, \quad r_2(u) = \tilde{u} - (x-1) \text{ for } t=0$$

$$\mathcal{V} = \{v \in H^1 \times C^1 \mid v(0,t) = v(1,t) = 0\}$$

$$\mathcal{V}_h = \left\{\sum_{i=1}^{N} c_i(t)\phi_i(x) \mid c_i \in C^1 \text{ and } \phi_i(x) \in H^1, \; \phi_i(0) = \phi_i(1) = 0\right\}$$ 0.5

For $t>0$

Find $c_j(t) \in C^{(0)}, j=1,\dots,N$: $\left(\sum d_i(t)\phi_i(x), \; r\left(\sum_{j=1}^{N} c_j(t)\phi_j(x)\right)\right) = 0$ for all $d_i(t) \in C^{(0)}, i = 1, \dots$

Find " $(\phi_i, r(\;\;)) = 0$ for all ...

" $$\sum \underbrace{(\phi_i, e^{-x}\phi_j)}_{M_{ij}}\frac{dc_j}{dt} - \sum \underbrace{\left(\phi_i, \frac{\partial}{\partial x}\left(\cos x \frac{\partial}{\partial x}\phi_j\right)\right)}_{A_{ij}} c_j + \underbrace{(\phi_i, \sin(x))}_{-b_i} = 0$$ 0.3

$$M\frac{dc}{dt} = Ac + b \quad t>0$$

3) 2

For $k=0$

Find $c_j^{(0)}$: $\left(\phi_i , r_2\left(\sum_{j=1}^{N} c_j^{(0)}\phi_j(x)\right)\right) = 0$

$$\sum_{j=1}^{N} \underbrace{(\phi_i, \phi_j)}_{N_{ij}} c_j^{(0)} - \underbrace{(\phi_i, x-1)}_{\hat{b}_i} = 0$$

$$N c = \hat{b}$$

0.3

1st b) The difference/Fourier method to get estimates of the relevant eigenvalues for the test equation can only be computed for differences with constant coefficients. Here the coefficients vary. A common way out is to replace the coefficients by constants representing the worst case. In this case, e^{-x} is replaced by e^{-1} and $\cos x$ by 1.

Suppose in the previous we use linear elements with constant size h. Give the typical row of M and A

Answer

$$M_{i,i+1} = \frac{1}{e}(\phi_i, \phi_{i+1}) = \frac{1}{e}\int_{x_i}^{x_{i+1}} \frac{x-x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x-x_i}{x_{i+1}-x_i} \, dx$$

$$= \frac{-1}{e}\frac{1}{h^2}\int_0^h (x-h)\, x \, dx = -\frac{1}{e}\frac{1}{h^2}\left(\frac{1}{3}x^3 - \frac{1}{2}hx^2\Big|_0^h\right) = \frac{h}{6e}$$

$$M_{i,i} = \frac{1}{e}(\phi_i, \phi_i) = \frac{1}{e}\left[\int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{x_i - x_{i-1}}\right)^2 dx + \int_{x_i}^{x_{i+1}} \cdots \, dx\right]$$

$$= \frac{1}{h^2}\frac{2}{e}\int_0^h x^2 dx = \frac{2}{3e} h$$

$M_{i,i-1} = \frac{h}{6e}$ due to symmetry

0.3

3
3

$$A_{i,i-1} = \left(\phi_i, \frac{\partial^2}{\partial x^2}\phi_{i-1}\right) \overset{\substack{\text{P.I.}\\ \phi(0)=\phi(1)=0}}{=} -\left(\frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_{i-1}}{\partial x}\right)$$

0.5

$$= -\int_{x_{i-1}}^{x_i} \frac{1}{h}\,\frac{-1}{h}\,dx = \frac{1}{h}$$

$$A_{i,i} = -\left(\frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_i}{\partial x}\right) = -2\int_{x_{i-1}}^{x_i}\left(\frac{1}{h}\right)^2 dx = -\frac{2}{h}$$

$$A_{i,i+1} = \frac{1}{h}$$

Eigen value problem

$$\lambda \frac{h}{6e}\left[1 \quad 4 \quad 1\right] e^{ij\varphi} = \frac{1}{h}\left[1 \quad -2 \quad 1\right] e^{ij\varphi}$$

$$\lambda \frac{h}{6e}\left[e^{-i\varphi} + 4 + e^{i\varphi}\right] = \frac{1}{h}\left[e^{-i\varphi} - 2 + e^{i\varphi}\right]$$ 0.4

$$\lambda = \frac{6e}{h^2}\,\frac{-2+2\cos\varphi}{4+2\cos\varphi} = \frac{6e}{h^2}\,\frac{-1+\cos\varphi}{2+\cos\varphi}$$

$\frac{1}{2}$ d Determine the interval containing these eigenvalues.

$$\min_{-1\leq x\leq 1} \frac{6e}{h^2}\,\frac{-1+x}{2+x}$$

differentiate and look for zero

$(2+x)-(-1+x) = 0$

$3 = 0$

monotonic increasing function

0.5

So $x=-1$ gives min and $x=1$ max

$$\lambda \in \frac{6e}{h^2}[-2, 0]$$

$\frac{1}{2}$ e $|1+z| < 1$ and z real negative $\rightarrow$ $-z-1 < 1$

$-z < 2$

$-\Delta t\lambda < 2$ 0.5

$\Delta t \frac{12e}{h^2} < 2$

$\Delta t \leq \frac{1}{6}\frac{h^2}{e}$